Level spacing distribution of pseudointegrable billiard

T.Shigehara

Computer Centre, University of Tokyo, Bunkyo-ku Yayoi, Tokyo 113, Japan

N.Yoshinaga

Department of Physics, College of Liberal Arts, Saitama University, Saitama 338, Japan

Taksu Cheon

Department of Physics, Hosei University, Fujimi, Chiyoda-ku, Tokyo 102, Japan

T. Mizusaki

Department of Physics, University of Tokyo, Bunkyo-ku Hongo, Tokyo 113, Japan

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Abstract

In this paper, we examine the level spacing distribution $P(S)$ of the rectangular billiard with a single point-like scatterer, which is known as pseudointegrable. It is shown that the observed $P(S)$ is a new type, which is quite different from the previous conclusion. Even in the strong coupling limit, the Poisson-like behavior rather than Wigner-like is seen for $S > 1$, although the level repulsion still remains in the small S region. The difference from the previous works is analyzed in detail.

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The *wave chaos* has recently attracted much attention as one of typical manifestations of quantum chaos. The wave chaos is a new class of chaos which is generated by purely quantal effects in the systems, the classical counterpart of which is non-chaotic. Many numerical and theoretical works have already been done on the pseudointegrable dynamical systems with a few degrees of freedom [1-17]. One of such systems is the two-dimensional rectangular billiard with a single point-like scatterer, for which the statistical properties of the eigenvalues and wave functions have been examined from various points of view [12-14]. In the recent paper [14], Šeba and Życzkowski examined this system throughout a wide range of energy excitation using the Green's function method and revealed some new aspects. One of their conclusions is that the level spacing distribution $P(S)$ becomes closer to the Wigner distribution in the strong coupling limit, although the fine structure does not conform to the prediction of GOE (Gaussian Orthogonal Ensemble). The appearance of Wigner-like distribution in $P(S)$ might be considered as further evidence which confirms the existence of the wave chaos.

There are now two approaches to the investigation of this pseudointegrable billiard. One of them is, as mentioned above, to analyze the Green's function of this system, and the other is to diagonalize the Hamiltonian by using the Fourier basis. While both approaches do not include any approximation formally, the truncation of the basis is inevitable in the actual numerical calculation. One should be most careful to decide the range of the basis because of the singularity of the interaction under consideration. In this brief paper, we will examine the effect of restriction of the basis on $P(S)$. This might sound strange because it might appear to be a mere technical problem in the numerical treatment lacking any physical interests. As we will see later, however, this is not the case and the closest care to the basis is essential for this system. In fact, the main conclusion is that the level spacing distribution $P(S)$ in the strong coupling limit never becomes Wigner-like but belongs to a quite new class.

More systematic analysis including the investigation of the other statistical properties than $P(S)$ concerning the eigenvalues and wave functions will be performed in the forthcom-

ing paper [18]. Here, we will restrict ourselves to revealing the influence of the limitation of the phase space on $P(S)$.

We follow the Green's function method. As the mathematical formalism is explained in full in [14], we only summarize those points that are necessary in the following discussion, stressing the physical aspects. The Hamiltonian of the two-dimensional rectangular billiard with a point-like scatterer is formally given by

$$H = -\frac{\Delta}{2M} + v_0 \delta(x - x_0)\delta(y - y_0), \tag{1}$$

where M is the mass of a particle, v_0 and (x_0, y_0) are strength and position of the scatterer respectively. The Green's function of this system is given by

$$G(x, y; x', y'; z) = G_F^{(0)}(x, y; x', y'; z)$$

$$+ G_F^{(0)}(x, y; x_0, y_0; z) \frac{v_0}{1 - G^{(0)}(x_0, y_0; x_0, y_0; z) v_0} G_F^{(0)}(x_0, y_0; x', y'; z). \tag{2}$$

Here, z is the energy variable, $G_F^{(0)}$ is the Green's function of the billiard without any scatterer and $G^{(0)}(x_0, y_0; x_0, y_0; z)$ describes the propagation of the particle which begins to propagate at the point-like scatterer and ends there. Clearly, the second term on the r.h.s. in Eq.(2) means the multiple scattering caused by the point-like scatterer. (Although the authors of [14] call $G^{(0)}(x_0, y_0; x_0, y_0; z)$ with opposite sign in Eq.(2) the meromorphic function $\xi(z)$, we call it just the Green's function in the following discussion as long as causing no confusion.) From Eq.(2), we see that, in the Green's function method, the eigenvalue problem is equivalent to solving the following equation,

$$G^{(0)}(x_0, y_0; x_0, y_0; z) = \frac{1}{v_0}. \tag{3}$$

Owing to the fact that the obstacle is point-like, the whole problem can be reduced to a transcendental equation instead of an integral equation. If the scatterer is located at the center of the rectangle, which is the case we will examine in this paper, the Green's function (meromorphic function) with the Dirichlet condition on the border of the billiard is given by

$$G^{(0)}(z) \equiv G^{(0)}(x_0, y_0; x_0, y_0; z) = \frac{4}{l_x l_y} \sum_{n_x,n_y=1}^{\infty} \{ \frac{1}{z - E^{(0)}_{2n_x-1,2n_y-1}} + \frac{E^{(0)}_{2n_x-1,2n_y-1}}{(E^{(0)}_{2n_x-1,2n_y-1})^2 + 1} \}, \quad (4)$$

$$E^{(0)}_{n_x,n_y} = \frac{\pi^2}{2M} \{ (\frac{n_x}{l_x})^2 + (\frac{n_y}{l_y})^2 \}. \quad (5)$$

Here, $E^{(0)}_{n_x,n_y}$ is the eigenvalue of the billiard without any obstacle and l_x and l_y are the side-lengths of the rectangle ($x_0 = l_x/2$ and $y_0 = l_y/2$). One should notice that, when the scatterer is placed at the center of the billiard, the scatterer affects only even-even parity states. The special feature in case of the singular interaction can be seen in the second term in the Green's function of Eq.(4), while the Green's function $G^{(0)}_F$ in the case without any obstacle in the billiard does not have the corresponding term. (The appearance of this term is closely related to the boundary condition around the scatterer. In order to determine its exact form, one needs the help of some theorems in the functional analysis. For details, see Ref. [14] and references therein.) One realizes that each of two terms in the Green's function $G^{(0)}(z)$ has logarithmic divergence when summed separately, although the sum of them leads to a finite value.

We examine the case that $M = 8\pi$, $l_x = \pi/3$ and $l_y = 3/\pi$. In this particular parameterization, the average density of even-even parity states is equal to one according to the Weyl's formula.

To see a general feature of the Green's function $G^{(0)}(z)$, the schematic graph of the Green's function is shown in Fig.1. Here, the eigenvalues $\{E^{(0)}_{2n_x-1,2n_y-1}\}$ of even-even parity states in the unperturbed system are renamed in ascending order as $\{E^{(0)}_n\}$. One can easily see that each eigenvalue E_n of the perturbed system is isolated between two unperturbed energies, $E^{(0)}_n$ and $E^{(0)}_{n+1}$, namely

$$E^{(0)}_1 < E_1 < E^{(0)}_2 < E_2 < E^{(0)}_3 < E_3 < \cdots,$$

and becomes larger as one increases the strength of the coupling. In the strong coupling limit ($v_0 = \infty$), the set of eigenvalues is just that of zeros of the Green's function.

In order to get the solutions of Eq.(3) numerically, one must limit the range of the summation with minimum and maximum values of n, n_{min} and n_{max},

$$G^{(0)}_{appro}(z) = 4 \sum_{n=n_{min}}^{n_{max}} \{ \frac{1}{z - E^{(0)}_n} + \frac{E^{(0)}_n}{(E^{(0)}_n)^2 + 1} \}. \tag{6}$$

The prescription of the limitation in [14] is to take $n_{min} = l - 500$ and $n_{max} = l + 500$ for looking for a root E_l of the Eq.(3) localized between $E^{(0)}_l$ and $E^{(0)}_{l+1}$. Hereafter, we quote this prescription as the *truncation (I)*. At first sight, this seems to be quite reasonable because the main contribution on the Green's function around the energy E_l comes from the terms that have n around l and because the contribution on the Green's function from $n < n_{min}$ tends to cancel that from $n > n_{max}$. According to the truncation (I), one gets the level spacing distribution $P(S)$ without much numerical labor. As a typical example, we show the case of the strong coupling limit in Fig.2. This corresponds to the Fig.2c in [14] and of course shows quite similar structure, although the parameters for the system are slightly different in both calculations. One might conclude from Fig.2 that the level spacing distribution of the rectangular billiard with a point-like scatterer is almost Wigner-like in the strong coupling limit.

We now examine the accuracy of the truncation (I). Fig.3 shows the same calculation as above except that $n_{min} = 1$ and $n_{max} = 100000$. Hereafter, We quote this case as the *truncation (II)*. We have numerically checked sufficient convergence of the eigenvalues in the energy region under consideration. Also, we will later justify this truncation of the basis in an analytic manner in this paper. One easily sees the drastic changes even in a qualitative level. For $S > 1$, $P(S)$ is rather Poisson-like than Wigner-like, although the level repulsion still remains in the small S region. The level repulsion is regarded as a common feature among the various pseudointegrable systems [10,15,17]. Roughly speaking, one might say that $P(S)$ in Fig.3 shows an *intermediate* feature between the regularity and the chaos.

In order to clarify the reason for the disagreement between the level spacing distributions in Fig.2 and Fig.3, we estimate the numerical error in the Green's function related to the truncation of the basis. The numerical error comes from the terms which are neglected by

the limitation of summation in Eq.(6),

$$\delta G^{(0)}(z) \equiv G^{(0)}(z) - G^{(0)}_{appro}(z) = 4(\sum_{n=1}^{n_{min}-1} + \sum_{n=n_{max}+1}^{\infty})\{\frac{1}{z - E_n^{(0)}} + \frac{E_n^{(0)}}{(E_n^{(0)})^2 + 1}\}. \tag{7}$$

To estimate the order of magnitude of error, we consider the unperturbed energy $E_n^{(0)}$ as the continuous variable and replace the summation by the integral as

$$\delta G^{(0)}(z) \simeq 4(\int_0^{E_{n_{min}}} + \int_{E_{n_{max}}}^{\infty})(\frac{1}{z-E} + \frac{E}{E^2+1})dE. \tag{8}$$

Here, one should notice that the mean level density is constant and equal to one in our parameterization. The integral in Eq.(8) is elementary and leads to

$$\delta G^{(0)}(z) \simeq 4(F(z, E_{n_{min}}) - F(z, 0) - F(z, E_{n_{max}})), \tag{9}$$

where the function F is defined by

$$F(z, E) = \frac{1}{2} \log \frac{E^2+1}{(z-E)^2}. \tag{10}$$

If $1 \ll E_{n_{min}} < z < E_{n_{max}}$ and $z \simeq \frac{E_{n_{min}} + E_{n_{max}}}{2}$, then one obtains

$$\delta G^{(0)}(z) \simeq 4(\log z + \log \frac{E_{n_{min}}}{E_{n_{max}}}). \tag{11}$$

Notice that the first term $\log z$ comes from $F(z, 0)$. This shows that if one evaluates, for example, $E_{1000}(\simeq 1000)$ according to the truncation (I), the numerical error

$$\delta G^{(0)}(1000) \simeq 4(\log 1000 + \log \frac{500}{1500}) \simeq 4(6.90 - 1.09) \simeq 23.2,$$

is accompanied. We have numerically checked the validity of this estimate of the error. Also, Eq.(11) shows that the error is much larger as one increases the energy. Clearly, the underestimation of the Green's function leads to the underestimation of the eigenvalues.

The accuracy of zeros is not directly related to the magnitude of the error in the Green's function, but to the ratio between the magnitude of the error and the derivative of the Green's function at the zero. Therefore, we further examine the derivative of the Green's function. As a typical example, we show in Table 1 some eigenvalues around E_{1000} obtained

by the truncation (I) and the derivative of the Green's function at the corresponding zero. For comparison, we show the result with the truncation (II). It can be easily seen from Table 1 that whereas the zero has a fairly good accuracy if the derivative there is large enough compared to the error (about 20), this is not the case if the derivative is small. In fact, some eigenvalues have numerical errors comparable to the mean energy difference between the nearest neighboring levels. It is also unfortunate for the truncation (I) that the sequence of the absolute value of the derivatives looks like random. So, the accuracy of a zero just next to a very accurate one can be very poor. This of course has a serious influence on $P(S)$.

On the contrary, the numerical error by the truncation (II) is given by

$$\delta G^{(0)}(z) \simeq -4F(z, E_{n_{max}}), \tag{12}$$

and quite small even for $E_{4000} \simeq 4000$

$$\delta G^{(0)}(4000) \simeq -\frac{4z}{E_{n_{max}}} \simeq -0.16. \tag{13}$$

Also, one can see that the large magnitude of the derivatives of the Green's function ensures the accuracy of the zeros. The absolute value of the typical error with a zero is estimated to be at most of the order of 10^{-3}, namely 0.1% compared to the mean level spacing.

The physical reason why such large phase space is necessary is obvious. It is the singularity of the interaction between the unperturbed levels. In fact, any pair of even-even parity states couples to each other with the same coupling strength, because the scatterer point is located at the center in the billiard. The singularity of the interaction is a common feature to certain kinds of the pseudointegrable systems. Extreme care in numerical accuracy is required in order to analyze such systems.

The reminiscence of the Poisson-like behavior (regularity) for $S > 1$ in the strong coupling limit is somewhat surprising. Although the reason for that is one of the most exciting topics, it goes beyond our present scope and remains as a future problem. We hope that our preliminary study in this paper serves as a guidepost leading to the right direction for studying the rich field of the new class of quantum chaos.

In summary, we have shown that the level spacing distribution $P(S)$ of the rectangular billiard with a point-like scatterer in the strong coupling limit belongs to a new class. Contrary to the previous conclusion, it does not show Wigner-like behavior, but shows Poisson-like for $S > 1$, although there remains the level repulsion in the small S region. A wide range of the Fourier basis is demanded in order to get the correct eigenvalues of this system.

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REFERENCES

[1] A.N.Zemlyakov and A.B.Katok, Math. Notes **18**, 291(1975).

[2] P.J.Richens and M.V.Berry, Physica D **2**, 495(1981).

[3] F.S.Henyey and N.Pomphrey, Physica D **6**, 78(1982).

[4] M.V.Berry and M.Wilkinson, Proc. R. Soc. London, Ser.A **392**, 15(1984).

[5] B.Eckhardt, J.Ford and F.Vivaldi, Physica D **13**, 339(1984).

[6] E.Gutkin, Physica D **19**, 311(1986).

[7] T.Cheon and T.D.Cohen, Phys. Rev. Lett. **62**, 2769(1989).

[8] B.Mirbach and H.J.Korsch, Nonlinearity **2**, 327(1989).

[9] J.H.Hannay and R.J.McCraw, J. Phys. A **23**, 887(1990).

[10] C.H.Lewenkopf, Phys. Rev. A **42**, 2431(1990).

[11] D.Biswas and S.R.Jain, Phys. Rev. A **42**, 3170(1990).

[12] P.Šeba, Phys. Rev. Lett. **64**, 1855(1990).

[13] T.Cheon, T.Mizusaki, T.Shigehara and N.Yoshinaga, Phys. Rev. A **44**, R809(1991).

[14] P.Šeba and K.Życzkowski, Phys. Rev. A **44**, 3457(1991).

[15] A.Shudo and Y.Shimizu, preprint, Waseda University, Tokyo, 1991.

[16] K.Życzkowski, Acta Phys. Pol. B **23** 245(1992).

[17] K.Życzkowski and P.Šeba, preprint.

[18] T.Shigehara, N.Yoshinaga, T.Cheon and T.Mizusaki, in preparation.

Figure Captions

Figure 1: Schematic graph of the Green's function in Eq.(4).

Figure 2: Level spacing distribution $P(S)$ in case of the strong coupling limit according to the truncation (I); $n_{min} = l - 500$ and $n_{max} = l + 500$ in Eq.(6). Statistics are taken within the eigenstates indicated in the figure. The Wigner (solid line) and Poisson (broken line) are also shown for reference.

Figure 3: Same as Fig.2 except that according to the truncation (II); $n_{min} = 1$ and $n_{max} = 100000$ in Eq.(6).

TABLES

TABLE I. Zeros and the derivatives of the Green's function at the corresponding zeros. The second and third columns show the results according to the truncation (I), whereas the fourth and fifth columns in case of the truncation (II).

n	E_n	$\lvert(G^{(0)}_{appr})'(E_n)\rvert$	E_n	$\lvert(G^{(0)}_{appr})'(E_n)\rvert$
995	994.29	49	994.52	251
996	995.60	23	996.03	169
997	996.55	89	996.71	292
998	997.30	95	997.43	365
999	999.39	12	1000.13	110
1000	1000.46	782	1000.49	1032
1001	1001.95	16	1002.44	178
1002	1003.50	15	1004.24	111
1003	1004.75	77	1005.00	159
1004	1005.31	618	1005.34	873
1005	1006.32	30	1006.70	171

Fig.1

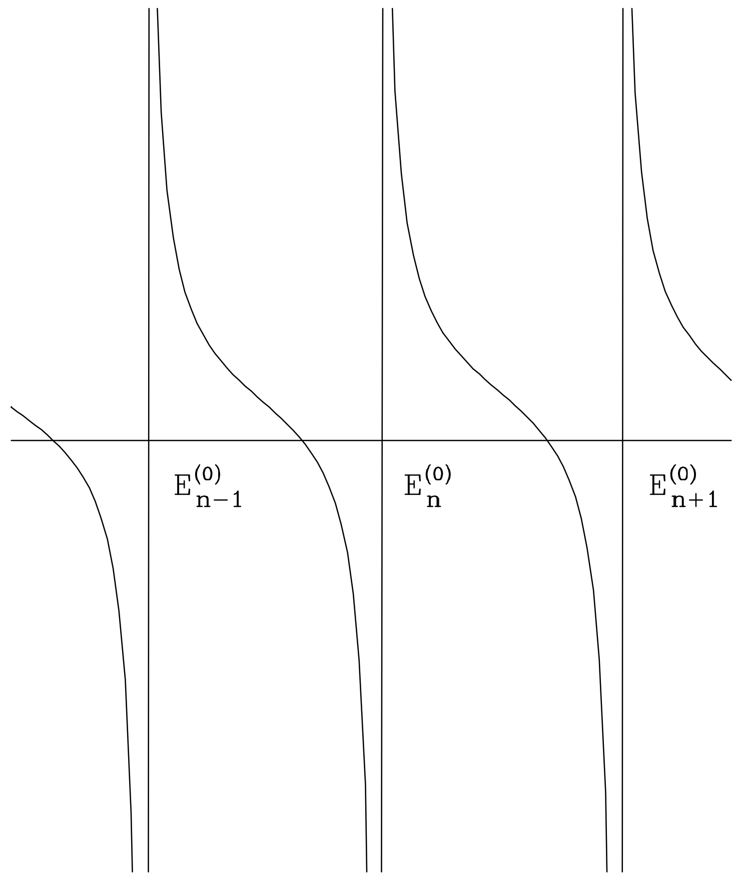


Fig.2

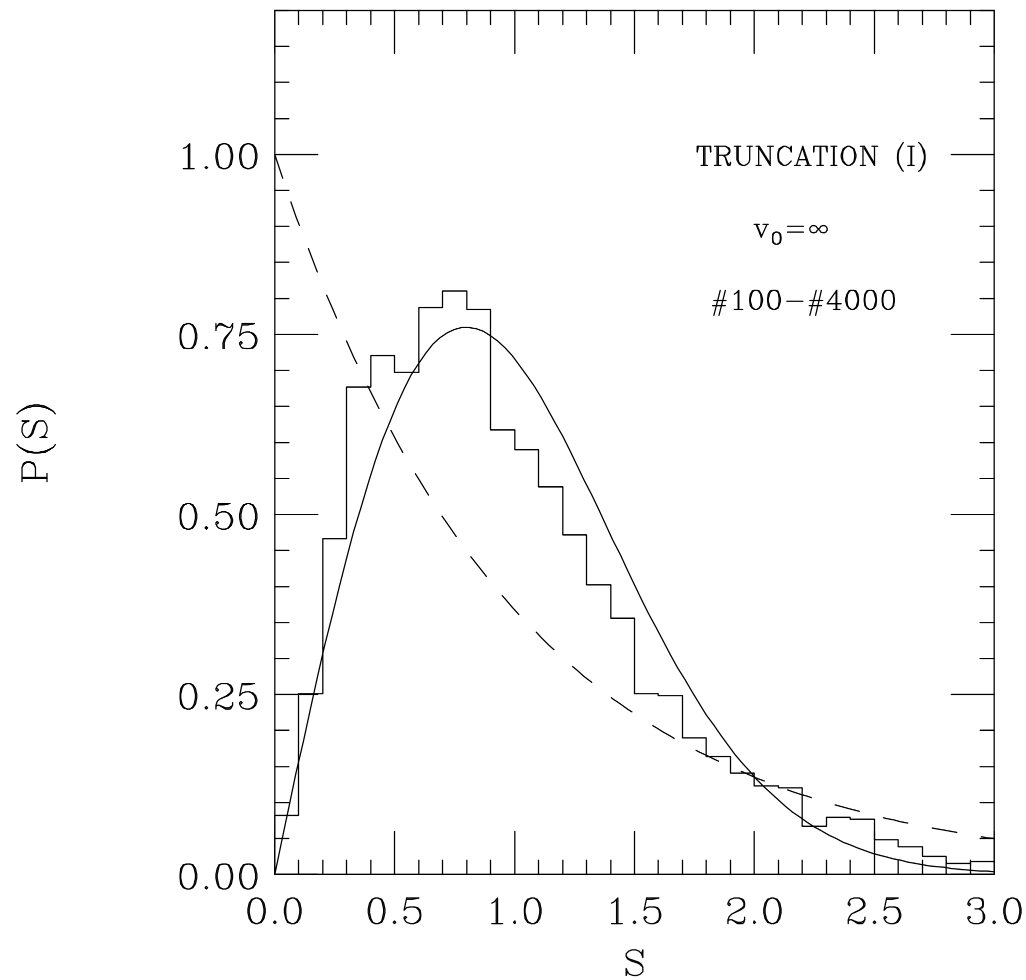
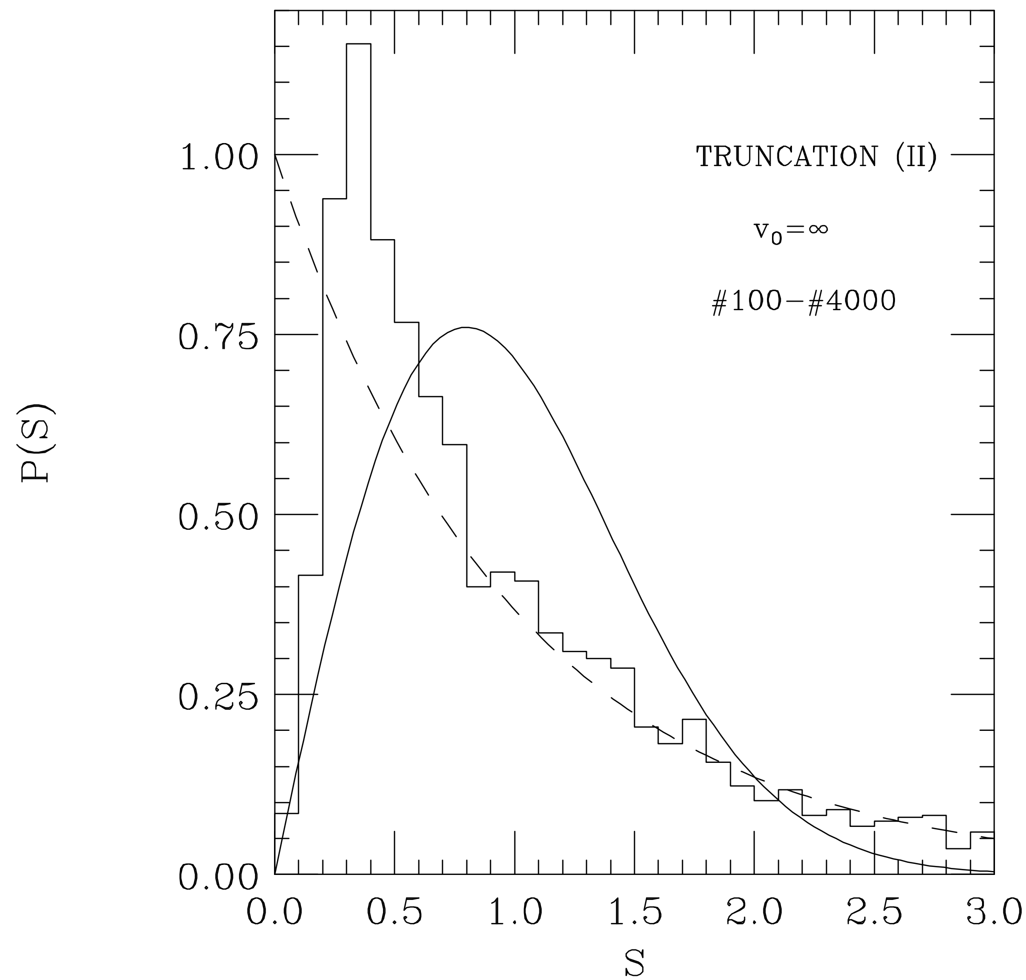

TRUNCATION (I)
$v_0=\infty$
#100−#4000
P(S)
S
1.00
0.75
0.50
0.25
0.00
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Fig.3



TRUNCATION (II)
$v_0=\infty$
#100−#4000
P(S)
S
0.00
0.25
0.50
0.75
1.00
0.0
0.5
1.0
1.5
2.0
2.5
3.0